MATH4010 Functional Analysis (2020-21): Homework 2: Deadline: 5 OCT 2020

Important Notice:

♣ The answer paper must be submitted before the deadline.
♠ The answer paper MUST BE sent to the CU Blackboard. Please refer to the course web for details.
✠ Each answer paper must include your name and student ID.

1. Let $X := \{f : [a,b] \to \mathbb{R} : f \quad \text{is continuous on} \quad \mathbb{R}\}$. For each $f \in X$, let $\|f\|_1 := \int_a^b |f(t)|dt$ and $\|f\|_\infty := \sup\{|f(t)| : t \in [a,b]\}$. Put

$$Tf(x) := \int_a^x f(t)dt$$

for $x \in [a,b]$.

   (i) Are $\|\cdot\|_\infty$ and $\|\cdot\|_1$ equivalent norms?

   (ii) Is X a Banach space under the norm $\|\cdot\|_1$?

   (iii) If T is viewed as the operator from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_\infty)$, find $\|T\|$.

   (iv) If T ie viewed as the operator from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_1)$, find $\|T\|$.

2. Let $(X, \|\cdot\|)$ be a normed space. Suppose that $X = E \oplus F$ for some closed subspaces E and F of X, that is, $E \cap F = \{0\}$ and for each $x \in X$, there are elements $u \in E$ and $v \in F$ such that $x = u + v$. We define a new norm on X by $\|x\|_1 := \|u\| + \|v\|$, where $x \in X$ and $x = u + v$ for some $u \in E$ and $v \in F$.

   Write X_1 for the normed space $(X, \|\cdot\|_1)$. Define a linear map

$$T : x \in E \mapsto \bar{x} \in X_1/F$$

   where $\bar{x}$ denotes the equivalence class of x in X_1/F. Show that T is a bounded linear isomorphism. Is the inverse T^{-1} also bounded?

$* * *$ **End** $* * *$